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Noncommutative Supersymmetric Gauge Anomaly

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ABSTRACT

We extend the general method of [4] to compute the consistent gauge anomaly for noncommutative 4d SSYM coupled to chiral matter. The choice of the minimal homotopy path allows us to obtain a simple and compact result. We perform the reduction to components in the WZ gauge proving that our result contains, as lowest component, the bosonic chiral anomaly for noncommutative YM theories recently obtained in literature.

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1 Introduction

In the recent literature a large number of papers have appeared devoted to various aspects of noncommutative field theory. Some of these papers have concerned themselves with noncommutative supersymmetric field theories, and a few have treated the subject in a superspace context [1]. In this work, we address ourselves to the superspace computation of the consistent anomaly for NCSYM, using some recently developed methods [2, 3, 4]. We obtain the noncommutative version of the result in ref. [4] and we show that the reduction to components agrees with the consistent anomaly for the bosonic NCYM recently computed in ref. [5].

2 NC super Yang–Mills coupled to chiral matter

The noncommutative $N = 1$ supersymmetric Yang–Mills theory in four dimensions can be defined on a superspace described by bosonic noncommutative x^μ coordinates, $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$ and spinorial anticommuting cooordinates $\{\theta^\alpha, \theta^\beta\} = \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = 0$ [1]. In chiral representation the constraints for the superspace covariant derivatives are solved by (we use the obvious generalization of *Superspace* [6] conventions)

$$\nabla_\alpha = (e^{-V})_\star \star D_\alpha (e^V)_\star \qquad \nabla_{\dot\alpha} = \bar D_{\dot\alpha} \qquad \nabla_a = -i\{\nabla_\alpha, \nabla_{\dot\alpha}\} \tag{2.1}$$

and the corresponding field strengths are given by

$$\begin{aligned} W_\alpha &= i\bar D^2 \left((e^{-V})_\star \star D_\alpha (e^V)_\star \right) \equiv \bar D^2 \Gamma_\alpha \\ \overline{W}_{\dot\alpha} &= i D^2 \left((e^{V})_\star \star \bar D_{\dot\alpha} (e^{-V})_\star \right) \equiv D^2 \overline{\Gamma}_{\dot\alpha} \end{aligned} \tag{2.2}$$

where Γ_α and $\overline{\Gamma}_{\dot\alpha}$ are the spinorial connections. Moreover, we have defined $(e^V)_\star \equiv 1 + V + \frac{1}{2} V \star V + \cdots$.

The dynamics of chiral scalar matter in the fundamental representation of the gauge group minimally coupled to the gauge field is described by the action

$$\int d^8 z \, \bar\Phi \star (e^V)_\star \star \Phi \tag{2.3}$$

where Φ and $\bar\Phi$ are chiral and antichiral superfields, respectively ($\bar D_{\dot\alpha} \Phi = D_\alpha \bar\Phi = 0$). Here we have used the notation $z \equiv (x, \theta, \bar\theta)$. This action is invariant under infinitesimal gauge transformations generated by (anti)chiral parameters Λ ($\bar\Lambda$)

$$\begin{aligned} \delta\phi &= i\Lambda \star \phi \qquad \delta\bar\phi = -i\bar\phi \star \bar\Lambda \\ \delta(e^V)_\star &= i\left(\bar\Lambda \star (e^V)_\star - (e^V)_\star \star \Lambda \right) \end{aligned} \tag{2.4}$$

In order to perform standard functional calculations we recall some basic identities satisfied by the $\star$–product. Using the following property

$$\int d^4x\,(f\star g)(x) \;=\; \int d^4x\, f(x)\cdot g(x) \tag{2.5}$$

the functional derivative in the noncommutative case can be defined as usual

$$\begin{aligned}\frac{\delta}{\delta\phi(y)}\int d^4x\,(\phi_1\star\phi_2\star\phi_3)(x) \;&=\; \int d^4x\,\frac{\delta\phi_1(x)}{\delta\phi_1(y)}\cdot(\phi_2\star\phi_3)(x)\\ =\; \int d^4x\,\delta^{(4)}(x-y)\,(\phi_2\star\phi_3)(x) &= (\phi_2\star\phi_3)(y)\end{aligned} \tag{2.6}$$

The same definition holds for the functional derivative with respect to ϕ_2 or ϕ_3 since, by the cyclicity of the $\star$ –product, the field with respect to which we differentiate can be written as the first entry of the product. As a consequence of the previous results we have for instance

$$\int d^4y\delta\phi_1(y)\frac{\delta}{\delta\phi_1(y)}\int d^4x\,(\phi_1\star\phi_2\star\phi_3)(x) \;=\; \int d^4y\,(\delta\phi_1\star\phi_2\star\phi_3)(y) \tag{2.7}$$

and more generally

$$\delta\int d^4x\,\phi_1\star\phi_2\star\phi_3 \;=\; \int d^4x\,(\delta\phi_1\star\phi_2\star\phi_3 \;+\; \phi_1\star\delta\phi_2\star\phi_3 \;+\; \phi_1\star\phi_2\star\delta\phi_3) \tag{2.8}$$

The generalization of the previous identities to superspace is straightforward. In particular, the functional derivative with respect to chiral and antichiral superfields is defined as usual

$$\frac{\delta\Phi(z)}{\delta\Phi(z')} \;=\; \bar{D}^2\delta^{(8)}(z-z') \qquad\qquad \frac{\delta\bar{\Phi}(z)}{\delta\bar{\Phi}(z')} \;=\; D^2\delta^{(8)}(z-z') \tag{2.9}$$

We now compute the covariant propagator for the chiral scalar described by the action (2.3). We write the functional integral

$$\left\langle\bar{\Phi}(z)\right\rangle = \int D\Phi D\bar{\Phi}\exp[-\int d^8w\;\bar{\Phi}\star(e^V)_\star\star\Phi]\cdot\bar{\Phi}(z) \tag{2.10}$$

and make a change of variable $\bar{\Phi}\to\bar{\Phi}+\delta\bar{\Phi}$ under which the functional integral doesn't change. One gets then

$$\begin{aligned}0 \;=\; \int D\Phi D\bar{\Phi}\exp[-\int d^8w\;(\bar{\Phi}\star(e^V)_\star\star\Phi)(w)]\\ \times[-\int d^8u\;(\delta\bar{\Phi}\star(e^V)_\star\star\Phi)(u)\cdot\bar{\Phi}(z)+\delta\bar{\Phi}(z)]\end{aligned} \tag{2.11}$$

Now taking the functional derivative $\delta/\delta\bar{\Phi}(z')$ and reinterpreting the functional integral as giving an expectation value, we obtain

$$\left\langle\int d^8uD^2\delta^{(8)}(u-z')\star((e^V)_\star\star\Phi)(u)\cdot\bar{\Phi}(z) - D^2\delta^{(8)}(z-z')\right\rangle \;=\; 0 \tag{2.12}$$

or

$$\left\langle \int d^8u \delta^{(8)}(u-z') \star D^2\Big[((e^V)_\star \star \Phi)(u) \cdot \bar{\Phi}(z)\Big] \right\rangle - D^2\delta^{(8)}(z-z') \;=\; 0 \qquad (2.13)$$

Using (2.6) we can write the previous equation as

$$D^2(e^V)_\star \star \left\langle \Phi(z') \; \bar{\Phi}(z) \right\rangle \;=\; D^2\delta^{(8)}(z'-z) \qquad (2.14)$$

where the $\star$–product on the l.h.s. is with respect to the z' variable. One now proceeds as in the commutative case: $\star$–multiplying by $(e^{-V})_\star$ on the left we have

$$\nabla^2 \star \left\langle \Phi(z') \; \bar{\Phi}(z) \right\rangle \;=\; (e^{-V})_\star \star D^2\delta^{(8)}(z'-z) \;=\; \nabla^2 \star (e^{-V})_\star \star \delta^{(8)}(z'-z) \qquad (2.15)$$

where $\nabla^2 \equiv \frac{1}{2}\nabla^\alpha \star \nabla_\alpha$. We proceed then multiplying by $\overline{\nabla}^2$ (remember that $\overline{\nabla}^2 = \bar{D}^2$, so the product is the standard product) and extending $\overline{\nabla}^2\nabla^2$ acting on the chiral superfield to the invertible operator

$$\Box_+ \;\equiv\; \nabla^2\overline{\nabla}^2 \;+\; \overline{\nabla}^2\nabla^2 \;-\; \overline{\nabla}^{\dot\alpha}\nabla^2\overline{\nabla}_{\dot\alpha} \;=\; \Box \;-\; iW^\alpha \star \nabla_\alpha \;-\; \frac{i}{2}(\nabla^\alpha \star W_\alpha) \qquad (2.16)$$

where

$$\Box \equiv \frac{1}{2}\nabla^{\alpha\dot\alpha} \star \nabla_{\alpha\dot\alpha} \qquad (2.17)$$

We obtain the covariant scalar propagator (we interchange z and z')

$$\left\langle \Phi(z) \; \bar{\Phi}(z') \right\rangle = \overline{\nabla}^2 \frac{1}{\Box_+} \star \nabla^2 \star (e^{-V})_\star \star \delta^{(8)}(z-z') \qquad (2.18)$$

where we have defined the inverse $1/\Box_+$ as the operator such that $1/\Box_+ \star \Box_+ \star f = \Box_+ \star 1/\Box_+ \star f = f$, for any f. Moreover, we have used the identity $\overline{\nabla}^2\Box_+ = \Box_+\overline{\nabla}^2$.

The short–distance behaviour of the propagator can be covariantly regularized [2] by introducing an UV cut–off M and multiplying it by $(\exp{(\Box_+/M^2)})_\star$

$$\left\langle \Phi(z) \; \bar{\Phi}(z') \right\rangle_{\rm reg} = (\exp{(\Box_+/M^2)})_\star \star \frac{1}{\Box_+} \star \overline{\nabla}^2\nabla^2 \star (e^{-V})_\star \star \delta^{(8)}(z-z') \qquad (2.19)$$

Now, using the obvious identities

$$d(e^{\Box_+ t})_\star \;=\; dt\Box_+ \star (e^{\Box_+ t})_\star \;=\; (e^{\Box_+ t})_\star \star \Box_+ dt \qquad (2.20)$$

we have

$$(e^{\Box_+/M^2})_\star \;=\; -\int_{\frac{1}{M^2}}^{\infty} d(e^{\Box_+ t})_\star \;=\; -\int_{\frac{1}{M^2}}^{\infty} dt (e^{\Box_+ t})_\star \star \Box_+ \qquad (2.21)$$

Multiplying on the right by $1/\Box_+$ we finally obtain

$$(e^{\Box_+/M^2})_\star \star \frac{1}{\Box_+} \;=\; -\int_{\frac{1}{M^2}}^{\infty} dt (e^{\Box_+ t})_\star \qquad (2.22)$$

An alternative expression for the regularized propagator is then

$$\left\langle \Phi(z) \; \bar{\Phi}(z') \right\rangle_{\rm reg} = -\int_{\frac{1}{M^2}}^{\infty} dt (e^{\Box_+ t})_\star \star \overline{\nabla}^2\nabla^2 \star (e^{-V})_\star \star \delta^{(8)}(z-z') \qquad (2.23)$$

3 The noncommutative consistent anomaly

To compute the consistent anomaly due to chiral matter coupled to supersymmetric noncommutative Yang–Mills, we follow the general procedure used in Refs. [3, 4] supplemented by the choice of a minimal homotopic path, as suggested in [7].

We introduce a homotopic path $g(y,V)$, $y \in [0,1]$, satisfying the boundary conditions $g(0,V) = 1$ and $g(1,V) = (e^V)_\star$ for any V. Moreover, we define the inverse path g^{-1} as given by $g^{-1} \star g = g \star g^{-1} = 1$. The extended superspace covariant derivatives and the corresponding spinorial field strengths

$$\begin{array}{rclrclrcl} \nabla_\alpha &=& g^{-1} \star D_\alpha g & \nabla_{\dot\alpha} &=& \bar D_{\dot\alpha} & \nabla_a &=& -i\{\nabla_\alpha, \nabla_{\dot\alpha}\} \\ \mathcal{W}_\alpha &=& i\bar D^2 (g^{-1} \star D_\alpha g) & \overline{\mathcal{W}}_{\dot\alpha} &=& iD^2(g \star \bar D_{\dot\alpha} g^{-1}) & & & \end{array} \tag{3.1}$$

satisfy the usual supersymmetry algebra. The homotopically extended action is given by

$$S = \int d^8 z \; \bar\Phi \star g \star \Phi \tag{3.2}$$

We write the effective action as

$$\Gamma[V] = \Gamma_{y=0}[V] + \int_0^1 dy\, \partial_y \Gamma = \int_0^1 dy \int d^8 z \; \partial_y g(z)_{ij} \star \left\langle \frac{\delta S}{\delta g(z)_{ij}} \right\rangle \tag{3.3}$$

or equivalently as

$$\Gamma[V] = \int_0^1 dy \int d^8 z \; \partial_y g(z)_{ij} \left\langle \frac{\delta S}{\delta g(z)_{ij}} \right\rangle \tag{3.4}$$

where we have set $\Gamma_{y=0}[V] = 0$. Using the explicit expression (3.2) we can formally write

$$\left\langle \frac{\delta S}{\delta g(z)_{ij}} \right\rangle = \Big\langle (\Phi \star \bar\Phi)(z) \Big\rangle_{ij} \equiv \lim_{z \to z'} e^{\frac{i}{2}\Theta\partial_x\partial_{x'}} \Big\langle \Phi(z)\; \bar\Phi(z') \Big\rangle_{ij} \tag{3.5}$$

where $\Theta\partial_x\partial_{x'} \equiv \Theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$. On the r.h.s. the extended propagator is present. Its expression is simply given by (2.18) where all occurences of $(e^V)_\star$ have been replaced by g. In particular, the covariant derivatives and the field strengths are the ones defined in (3.1).

Using the regularizations (2.19) or (2.23), the regularized expression for (3.5) is then given by

$$\begin{aligned} \left\langle \frac{\delta S}{\delta g(z)} \right\rangle &= \lim_{z' \to z} e^{\frac{i}{2}\Theta\partial_x\partial_{x'}} (\exp(\Box_+/M^2))_\star \star \frac{1}{\Box_+} \star \overline{\nabla}^2 \nabla^2 \star (e^{-V})_\star \star \delta^{(8)}(z-z') \\ &= -\lim_{z' \to z} e^{\frac{i}{2}\Theta\partial_x\partial_{x'}} \int_{\frac{1}{M^2}}^{\infty} dt (e^{\Box_+ t})_\star \star \overline{\nabla}^2 \nabla^2 \star (e^{-V})_\star \star \delta^{(8)}(z-z') \end{aligned} \tag{3.6}$$

To construct the consistent anomaly we consider the variation of Γ due to an infinitesimal gauge transformation on V. Due to the identities (2.5, 2.8), the formal derivation follows exactly the same steps as in the standard commutative case [4]. Varying (3.5), after an integration by parts on y we can write the consistent anomaly as

$$\begin{aligned}\delta\Gamma[V] \quad &\equiv \; L \;-\; \frac{1}{16\pi^2}\int_0^1 dy\, X(y) \\ &= \; \int d^8z\, \delta(e^V)_\star \left\langle \frac{\delta S}{\delta(e^V)_\star}\right\rangle \;+\; \int_0^1 dy \int d^8z \int d^8z''\, \delta g(z'')_{kl}\partial_y g(z)_{ij} \\ &\quad \times \left[\frac{\delta}{\delta g(z'')_{kl}}\left\langle\frac{\delta S}{\delta g(z)_{ij}}\right\rangle \;-\; \frac{\delta}{\delta g(z)_{ij}}\left\langle \frac{\delta S}{\delta g(z'')_{kl}}\right\rangle\right] \qquad (3.7)\end{aligned}$$

Note that, due to the identity (2.5), this expression can be equally well written with $\star$–products replacing the usual products. Being the variation of an effective action, the anomaly (3.7) automatically satisfies the WZ consistency condition.

We first concentrate on the explicit evaluation of the covariant term L. From the identity (2.19) we can write (we make repeated use of identity (2.5))

$$\begin{aligned} L \;&=\; \int d^8z \lim_{z'\to z}\, \delta(e^V)_\star \star e^{\frac{i}{2}\Theta\partial_x\partial_{x'}}(e^{\Box_+/M^2})_\star \star \frac{1}{\Box_+}\star \overline{\nabla}^2\nabla^2 \star (e^{-V})_\star \star \delta^{(8)}(z-z') \\ &=\; \int d^8z \lim_{z'\to z}\, (e^{-V})_\star \star \delta(e^V)_\star \star e^{\frac{i}{2}\Theta\partial_x\partial_{x'}}(e^{\Box_+/M^2})_\star \star \frac{1}{\Box_+}\star \overline{\nabla}^2\nabla^2\delta^{(8)}(z-z') \end{aligned} \qquad (3.8)$$

(a trace in group theory labels is understood) where we have moved the factor of $(e^{-V})_\star$ in front since we are dealing with a trace, both in group theory and superspace, and we have used the cyclicity property of the $\star$–product. From the identity

$$(e^{-V})_\star \star \delta(e^V)_\star \;=\; -\,i\left(\Lambda - (e^{-V})_\star \star \bar{\Lambda}\star (e^V)_\star\right) \;\equiv\; -\,i(\Lambda - \tilde{\Lambda}) \qquad (3.9)$$

it is easy to prove that the expression for the covariant term splits into the sum of a holomorphic and an anti-holomorphic contribution. We can then concentrate only on the holomorphic part, the $\tilde{\Lambda}$ term being simply obtained by hermitean conjugation.

Proceeding as in the commutative case, we replace in (3.8) $(e^{-V})_\star \star \delta(e^V)_\star$ with $-i\Lambda$, and first pull out a $\overline{\nabla}^2$ from the superspace measure which, due to the chirality property of the various quantities in the integrand, can only act on the z' appearing in the δ-function (recall that we are first taking the limit $z' \to z$). Using $\overline{\nabla}^2\nabla^2\delta^{(8)}(z-z')\overleftarrow{\overline{\nabla}^2} = \overline{\nabla}^2\nabla^2\overline{\nabla}^2\delta^{(8)}(z-z') = \Box_+\overline{\nabla}^2\delta^{(8)}(z-z')$ we obtain

$$L = -i\int d^6z \lim_{z'\to z} \Lambda \star e^{\frac{i}{2}\Theta\partial_x\partial_{x'}}(e^{\Box_+/M^2})_\star \overline{\nabla}^2\delta^{(8)}(z-z') \qquad (3.10)$$

Now, we write

$$\delta^{(8)}(z-z') = \frac{M^4}{(2\pi)^4} \int d^4k e^{iMk(x-x')} \delta^{(4)}(\theta-\theta') \tag{3.11}$$

and move the $\exp(iMkx)$ factor past the $(e^{\Box_+/M^2})_\star$ operator. Neglecting terms in the exponential which eventually do not contribute to the anomaly, we obtain

$$\begin{aligned} L &= -i\frac{M^4}{(2\pi)^4} \int d^4x d^2\theta \int d^4k\, \Lambda \star \lim_{x'\to x} e^{\frac{i}{2}\Theta\partial_x\partial_{x'}} e^{iMkx} e^{-iMkx'} \\ & \lim_{\theta'\to\theta} (e^{-k^2+ik^a\nabla_a/M+\Box/M^2-iW^\alpha\star\nabla_\alpha/M^2-i(\nabla^\alpha\star W_\alpha)/(2M^2)})_\star \overline{\nabla}^2 \delta^{(4)}(\theta-\theta') \end{aligned} \tag{3.12}$$

Now, we observe that

$$\lim_{x'\to x} e^{\frac{i}{2}\Theta\partial_x\partial_{x'}} e^{iMkx} e^{-iMkx'} \;=\; e^{iMkx} \star e^{-iMkx} \;=\; 1 \tag{3.13}$$

Moreover, performing the limit on the spinorial coordinates we obtain a zero result except from terms, in the expansion of the exponential, that can produce a factor of ∇^2 which together with the $\overline{\nabla}^2$ remove the $\delta^{(4)}(\theta-\theta')$. The only non–vanishing contribution comes from the second order term $1/2!(W^\alpha \star \nabla_\alpha)^2$ which also has the correct $1/M^4$ factor to cancel the overall M^4. Therefore, one can now take the limit $\theta' \to \theta$, remove the regulator, $M \to \infty$, perform the k-integration of the remaining e^{-k^2} factor, and obtain the final form of the covariant anomaly

$$L = -\frac{i}{8\pi^2} \int d^6z \ \mathrm{Tr}\Big[\Lambda \star W^\alpha \star W_\alpha\Big] \;+\; \mathrm{h.c.} \tag{3.14}$$

We now focus on the evaluation of the consistent term in (3.7). We consider the first contribution in that expression (the second one is simply obtained by interchanging δg with $\partial_y g$)

$$\begin{aligned} &\int_0^1 dy \int d^8z \int d^8z'' \delta g(z'')_{kl} \partial_y g(z)_{ij} \frac{\delta}{\delta g(z'')_{kl}} \left\langle \frac{\delta S_y}{\delta g(z)_{ij}} \right\rangle \\ = & -\int_0^1 dy \int d^8z \partial_y g(z) \lim_{z'\to z} e^{\frac{i}{2}\Theta\partial_x\partial_{x'}} \delta \int_{\frac{1}{M^2}}^\infty dt\, (e^{\Box_+ t})_\star \star \overline{\nabla}^2 \nabla^2 \star g^{-1} \star \delta^{(8)}(z-z') \end{aligned} \tag{3.15}$$

The calculation follows exactly the one for the commutative case, so that we list here only the main steps, referring to [4] for more details.

Starting from (3.15) we first perform the variation of the single operators. Using the identities

$$\delta g^{-1} \;=\; -g^{-1} \star \delta g \star g^{-1}$$

$$
\begin{aligned}
\delta\nabla_\alpha &= \delta(g^{-1}\star D_\alpha g) = [\nabla_\alpha, g^{-1}\star\delta g]_\star \\
\delta\nabla^2\star g^{-1} &= -g^{-1}\star\delta g\star\nabla^2\star g^{-1} \\
\delta(e^{\Box_+ t})_\star &= \int_0^t ds (e^{\Box_+ s})_\star\star\delta\Box_+\star(e^{\Box_+(t-s)})_\star \\
\delta\Box_+\Big|_{on\ chirals} &= \delta\overline{\nabla}^2\nabla^2 \;=\; \overline{\nabla}^2[\nabla^2, g^{-1}\star\delta g]_\star
\end{aligned} \tag{3.16}
$$

we obtain

$$
\begin{aligned}
&\int_0^1 dy\int d^8z\,\partial_y g(z)\lim_{z'\to z} e^{\frac{i}{2}\Theta\partial_x\partial_{x'}}\left[\int_{\frac{1}{M^2}}^\infty dt\,(e^{\Box_+ t})_\star\star\overline{\nabla}^2 g^{-1}\star\delta g\star\nabla^2\star g^{-1}\right. \\
&\left.-\int_{\frac{1}{M^2}}^\infty dt\int_0^t ds(e^{\Box_+ s})_\star\star\overline{\nabla}^2[\nabla^2, g^{-1}\star\delta g]_\star\star(e^{\Box_+(t-s)})_\star\star\overline{\nabla}^2\nabla^2\star g^{-1}\right]\star\delta^{(8)}(z-z')
\end{aligned} \tag{3.17}
$$

Here $[A,B]_\star$ is the Moyal bracket $[A,B]_\star = A\star B - B\star A$. Now, expanding the commutator in the second line and using the cyclicity properties of the trace and the $\star$–product, it is easy to see that the first term eventually cancels when added to the second contribution in (3.7) (the one obtained by interchanging δg with $\partial_y g$). The second term from the commutator gives instead a nontrivial contribution which we rewrite in the following way

$$
\begin{aligned}
&\int_{\frac{1}{M^2}}^\infty dt\int_0^t ds(e^{\Box_+ s})_\star\star\overline{\nabla}^2 g^{-1}\star\delta g\star\nabla^2\star(e^{\Box_+(t-s)})_\star\star\overline{\nabla}^2\nabla^2\star g^{-1} \\
&\qquad = \int_{\frac{1}{M^2}}^\infty dt\int_0^t ds(e^{\Box_+ s})_\star\star\overline{\nabla}^2 g^{-1}\star\delta g\star\nabla^2\star\overline{\nabla}^2(e^{\nabla^2\overline{\nabla}^2(t-s)})_\star\star\nabla^2\star g^{-1} \\
&\qquad = \int_{\frac{1}{M^2}}^\infty dt\int_0^t ds(e^{\Box_+ s})_\star\star\overline{\nabla}^2 g^{-1}\star\delta g\star\frac{\partial}{\partial t}(e^{\Box_-(t-s)})_\star\star\nabla^2\star g^{-1}
\end{aligned} \tag{3.18}
$$

where we have defined

$$
\Box_- \;=\; \nabla^2\overline{\nabla}^2 + \overline{\nabla}^2\nabla^2 - \nabla^\alpha\star\overline{\nabla}^2\nabla_\alpha \;=\; \Box \;-\; i\widetilde{\mathcal{W}}^{\dot\alpha}\star\overline{\nabla}_{\dot\alpha} \;-\; \frac{i}{2}(\overline{\nabla}^{\dot\alpha}\widetilde{\mathcal{W}}_{\dot\alpha}) \tag{3.19}
$$

with $\widetilde{\mathcal{W}}_{\dot\alpha} = -g^{-1}\star\overline{\mathcal{W}}_{\dot\alpha}\star g$. To obtain the expression (3.18) we have used the identity

$$
\nabla^2\overline{\nabla}^2\star(e^{\nabla^2\overline{\nabla}^2(t-s)})_\star\star \;=\; \frac{\partial}{\partial t}(e^{\nabla^2\overline{\nabla}^2(t-s)})_\star\star \tag{3.20}
$$

which can be easily proved by Taylor expanding the exponential.

The integration by parts of the ∂_t derivative produces an integrated term which cancels the first term in (3.17) and a second term which reads

$$
\begin{aligned}
&-\lim_{z'\to z} e^{\frac{i}{2}\Theta\partial_x\partial_{x'}}\int_0^1 dy\int d^8z\int_0^{\frac{1}{M^2}} ds \\
&\qquad \partial_y g(z)\star(e^{\Box_+ s})_\star\star\overline{\nabla}^2 g^{-1}\star\delta g\star(e^{\Box_-(1/M^2-s)})_\star\star\nabla^2\star g^{-1}\star\delta^{(8)}(z-z')
\end{aligned} \tag{3.21}
$$

This expression can be manipulated in the same manner as we treated the covariant anomaly L. We introduce a momentum basis for the $\delta^{(4)}(x-x')$ factor (see eq. (3.11)) and let the exponentials act on the e^{ikx} factor. This operation allows us to bring the $e^{ik(x-x')}$ term in front, while producing factors of Mk in the various exponentials. The $x' \to x$ limit can be performed now using the identity (3.13). In the limit $M^2 \to \infty$ and $\theta' \to \theta$ factors from the expansion of the exponentials proportional to $(\mathcal{W}^\alpha \star \nabla_\alpha)/M^2$, $(\nabla^\alpha \star \mathcal{W}_\alpha)/M^2$, $(\widetilde{\mathcal{W}}^{\dot\alpha} \star \overline{\nabla}_{\dot\alpha})/M^2$ and $(\overline{\nabla}^{\dot\alpha}\widetilde{\mathcal{W}}_{\dot\alpha})/M^2$ give the relevant contributions, cancelling the overall M^2 and the $\delta^{(4)}(\theta-\theta')$ factor. Adding the contribution obtained by $\delta g \leftrightarrow \partial_y g$, the final result takes the form

$$\begin{aligned} X(y) &= 2i\int_0^1 dy \int d^8z \,\mathrm{Tr}\, h_1 \star \Big([\mathcal{D}^\alpha h_2, \mathcal{W}_\alpha]_\star + \Big[\bar{\mathcal{D}}_{\dot\alpha} h_2, \widetilde{\mathcal{W}}^{\dot\alpha}\Big]_\star + \Big\{h_2, \bar{\mathcal{D}}_{\dot\alpha}\widetilde{\mathcal{W}}^{\dot\alpha}\Big\}_\star \Big) \\ &= -2i\int_0^1 dy \int d^8z \,\mathrm{Tr}\, \Big(h_1 \star \Big[\bar{\mathcal{D}}^{\dot\alpha} h_2, \widetilde{\mathcal{W}}_{\dot\alpha}\Big]_\star \; + \; h_2 \star [\mathcal{D}^\alpha h_1, \mathcal{W}_\alpha]_\star \Big) \\ &= -\frac{2}{3} i \int_0^1 dy \int d^8z \,\mathrm{Tr_s}\, \Big(h_1 \star (\bar{\mathcal{D}}^{\dot\alpha} h_2) \star \widetilde{\mathcal{W}}_{\dot\alpha} \; + \; h_2 \star (\mathcal{D}^\alpha h_1) \star \mathcal{W}_\alpha \Big) \end{aligned} \tag{3.22}$$

where $h_1 \equiv g^{-1} \star \delta g$ and $h_2 \equiv g^{-1} \star \partial_y g$. Moreover we have defined $\mathcal{D}_\alpha A \equiv \{\nabla_\alpha, A]_\star$ for any scalar or spinor object A in the adjoint representation of the gauge group. In the last equality $\mathrm{Tr_s}$ is the symmetrized trace defined as

$$\begin{aligned} \mathrm{Tr}_s(A \star B^\alpha \star C_\alpha) \equiv \mathrm{Tr}\Big[A \star \Big(B^\alpha \star C_\alpha - C_\alpha \star B^\alpha \Big) \\ + \; B^\alpha \star \Big(C_\alpha \star A + A \star C_\alpha \Big) \; - \; C_\alpha \star \Big(A \star B^\alpha + B^\alpha \star A \Big) \Big] \end{aligned} \tag{3.23}$$

for any scalar A and spinors B_α, C_α. Using the noncommutative extended Bianchi identities $\mathcal{D}^\alpha \mathcal{W}_\alpha + \bar{\mathcal{D}}^{\dot\alpha} \widetilde{\mathcal{W}}_{\dot\alpha} = 0$, one can easily show that the previous expression is antisymmetric under the exchange $h_1 \leftrightarrow h_2$.

As in the commutative case [3, 4], two different choices of homotopic paths lead to two cohomologically equivalent expressions for the consistent anomaly. Following Refs. [7, 4], we choose the 4D, noncommutative $N = 1$ supersymmetric Yang-Mills gauge theory "minimal" homotopy operator

$$g \;\equiv\; 1 \;+\; y(\;(e^V)_\star \;-\; 1\;) \tag{3.24}$$

The advantage of this choice is easily understood from the identities

$$\delta g \;=\; -\,iy\,\Big((e^V)_\star \star \Lambda - \bar\Lambda \star (e^V)_\star \Big) \qquad\qquad \partial_y g \;=\; \Big((e^V)_\star - 1 \Big) \tag{3.25}$$

which allow us to express h_1 and h_2 in (3.22) still as functions of $(e^V)_\star$. However, since in the covariant derivatives the inverse g^{-1} is present, the final expression for the noncommutative anomaly is necessarily a non–polynomial function of $(e^V)_\star$. This signals the presence of a no–go theorem analogue to the one proved for the standard commutative SYM [8].

With the minimal choice for the homotopy the 4D, noncommutative $N = 1$ supersymmetric Yang-Mills consistent anomaly is given by the imaginary part of a superaction, $\mathcal{A}_{\rm BGJ}(\Lambda, \bar{\Lambda}) = \mathcal{I}{\rm m}[\widetilde{\mathcal{A}}_{\rm BGJ}(\Lambda)\,]$ where

$$
\begin{aligned}
\widetilde{\mathcal{A}}_{\rm BGJ}(\Lambda) \;=\; & \frac{1}{4\pi^2}\int d^8z \Big\{ {\rm Tr}(\Lambda\Gamma^\alpha \star W_\alpha) \\
& -\; \frac{1}{3}\int_0^1 dy\, y\, {\rm Tr}_s\Big([(e^V)_\star \star \mathcal{G} \star \Lambda] \star \pi^\alpha \star \mathcal{W}_\alpha \\
& \qquad +\; [\, I \;-\; (e^V)_\star \star \mathcal{G}\,] \star [\,\tilde{\pi}^{\dot{\alpha}} \star \Lambda\,] \star \widetilde{\mathcal{W}}_{\dot{\alpha}}\Big)\Big\}
\end{aligned}
\tag{3.26}
$$

Here we have defined

$$
\begin{aligned}
\mathcal{G} &\equiv \Big[\, 1 \;+\; y\,(\,(e^V)_\star \;-\; 1\,)\,\Big]^{-1} \quad , \quad W_\alpha \;\equiv\; \mathcal{W}_\alpha(y=1) \;=\; \bar{D}^2\Gamma_\alpha \\
\pi_\alpha &\equiv\; (e^V)_\star \star \mathcal{G} \star \mathcal{G} \star \Gamma_\alpha \quad , \quad \tilde{\pi}_{\dot{\alpha}} \;\equiv\; e^V\, \mathcal{G} \star \tilde{\Gamma}_{\dot{\alpha}} \star \mathcal{G} \\
\mathcal{W}_\alpha &\equiv\; \bar{D}^2(\mathcal{G} \star D_\alpha \mathcal{G}^{-1}) \quad , \quad \widetilde{\mathcal{W}}_{\dot{\alpha}} \;\equiv\; \mathcal{G} \star [D^2(\mathcal{G}^{-1} \star \bar{D}_{\dot{\alpha}}\mathcal{G})\,] \star \mathcal{G}^{-1}
\end{aligned}
\tag{3.27}
$$

Explicitly, the extended field strengths are given by

$$
\begin{aligned}
\mathcal{W}_\alpha \;&\equiv\; y(e^V)_\star \star \mathcal{G} \star \omega_\alpha \\
\omega_\alpha \;&\equiv\; W_\alpha \;-\; (1-y)\Big[\tilde{\Gamma}^{\dot{\alpha}} \star \mathcal{G} \star \Gamma_a \;+\; (1-y)\tilde{\Gamma}^{\dot{\alpha}} \star \mathcal{G} \star \tilde{\Gamma}_{\dot{\alpha}} \star \mathcal{G} \star \Gamma_\alpha \\
& \qquad -\; \frac{i}{2}(\bar{D}^{\dot{\alpha}}\tilde{\Gamma}_{\dot{\alpha}} \;-\; i\tilde{\Gamma}^{\dot{\alpha}} \star \tilde{\Gamma}^{\dot{\alpha}}) \star \mathcal{G} \star \Gamma_\alpha\Big]
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
\widetilde{\mathcal{W}}_{\dot{\alpha}} \;&\equiv\; y(e^V)_\star \star \mathcal{G} \star \tilde{\omega}_{\dot{\alpha}} \\
\tilde{\omega}_{\dot{\alpha}} \;&\equiv\; \widetilde{W}_{\dot{\alpha}} \;+\; (1-y)\Big[\tilde{\Gamma}_a \star \mathcal{G} \star \Gamma^\alpha \;-\; (1-y)\tilde{\Gamma}_{\dot{\alpha}} \star \mathcal{G} \star \Gamma^\alpha \star \mathcal{G} \star \Gamma_\alpha \\
& \qquad -\; \frac{i}{2}(D^\alpha\Gamma_\alpha \;+\; i\Gamma^\alpha \star \Gamma_\alpha)\Big]
\end{aligned}
\tag{3.29}
$$

All the extended functions appearing on the l.h.s. of (3.27– 3.29) are expressed in terms of standard connections and field strengths of the NCSYM theory. The tilde quantities on the r.h.s. of these equations are defined as $\tilde{A} \equiv (e^{-V})_\star \star A \star (e^V)_\star$. In the derivation of (3.26–3.29) we have made repeated use of the identities

$$
D_\alpha(e^V)_\star \;=\; -\,i(e^V)_\star \star \Gamma_\alpha \quad , \quad \bar{D}_{\dot{\alpha}}(e^V)_\star \;=\; i(e^V)_\star \star \tilde{\Gamma}_{\dot{\alpha}}
\tag{3.30}
$$

which follow from the definitions (2.2), and

$$
\mathcal{G} \star (e^V)_\star \;=\; (e^V)_\star \star \mathcal{G}
\tag{3.31}
$$

$$
y\,\mathcal{G} \star (e^V)_\star \;=\; 1 \;-\; (1-y)\,\mathcal{G} \quad , \quad y\,((e^V)_\star - 1) \star \mathcal{G} \;=\; 1 \;-\; \mathcal{G}
\tag{3.32}
$$

which are a consequence of the definition of the inverse function $\mathcal{G}$.

4 The physical bosonic component

We perform the reduction to components of eq. (3.26) and evaluate explicitly the physical bosonic term. This amounts to compute the superspace integral and set fermions and auxiliary fields to zero (auxiliaries can be neglected since they would produce higher order contributions).

Since the noncommutativity features of the theory do not affect the superfield structure, the definition of the WZ gauge can be inherited from the commutative SYM theory [6]. Therefore, we perform the reduction by choosing the gauge

$$V| \;=\; DV| \;=\; D^2 V| \;=\; 0 \tag{4.1}$$

where "|" means evaluation at $\theta = \bar{\theta} = 0$. The basic ingredients required in the calculation are the physical bosonic components of the connections and the field strengths. From the identities

$$D_\alpha \, \bar{D}_{\dot\alpha} \, V| \;=\; -\, A_{\alpha\dot\alpha} \qquad\qquad \bar{D}_{\dot\alpha} \, D_\alpha \, V| \;=\; A_{\alpha\dot\alpha} \tag{4.2}$$

one easily obtains ($\rightarrow$ indicates that fermions and auxiliary fields have been neglected)

- Spinorial connections

$$\begin{aligned} &\bar{D}_{\dot\alpha} \Gamma_\alpha| \;=\; i A_{\alpha\dot\alpha} \qquad D_\beta \, \bar{D}^2 \, \Gamma_\alpha| \;\rightarrow\; f_{\alpha\beta} \\ &D_\alpha \, \bar{\Gamma}_{\dot\alpha}| \;=\; i A_{\alpha\dot\alpha} \qquad D^2 \, \bar{D}_{\dot\beta} \, \bar{\Gamma}_{\dot\alpha}| \;\rightarrow\; \bar{f}_{\dot\alpha\dot\beta} \;+\; \partial^\beta{}_{\dot\beta} \, A_{\beta\dot\alpha} \end{aligned} \tag{4.3}$$

- Vector connection

$$\begin{aligned} &\Gamma_a| \;=\; A_{\alpha\dot\alpha} \\ &D_\beta \bar{D}_{\dot\beta} \Gamma_a| \;\rightarrow\; -\, i \, C_{\dot\alpha\dot\beta} \, f_{\alpha\beta} \qquad D^\beta \, \bar{D}^2 \, D_\beta \, \Gamma_a| \;\rightarrow\; -\, \partial^\beta{}_{\dot\alpha} \, f_{\alpha\beta} \\ &\bar{\Gamma}_a| \;=\; A_{\alpha\dot\alpha} \\ &D_\beta \bar{D}_{\dot\beta} \bar{\Gamma}_a| \;\rightarrow\; i \, C_{\alpha\beta} \, \bar{f}_{\dot\alpha\dot\beta} \;+\; i \, \partial_{\beta\dot\beta} \, A_{\alpha\dot\alpha} \qquad D^\beta \, \bar{D}^2 \, D_\beta \, \bar{\Gamma}_a| \;\rightarrow\; -\, \partial_\alpha{}^{\dot\beta} \, \bar{f}_{\dot\alpha\dot\beta} \end{aligned} \tag{4.4}$$

- Field strengths

$$D_\beta \, W_\alpha| \;\rightarrow\; f_{\alpha\beta} \qquad\qquad \bar{D}_{\dot\alpha} \, \overline{W}_{\dot\beta}| \;\rightarrow\; \bar{f}_{\dot\alpha\dot\beta} \tag{4.5}$$

Using these identities we can compute the relevant bosonic components of the quantities which enter eq. (3.26). For ω_α and $\tilde{\omega}_{\dot\alpha}$ we find

$$\begin{aligned} &D_\beta \, \omega_\alpha| \;\rightarrow\; f_{\alpha\beta} \;-\; i(1-y) \, A_\beta{}^{\dot\alpha} \star A_{\alpha\dot\alpha} \\ &D^2 \, \bar{D}_{\dot\beta} \, \omega_\alpha| \;\rightarrow\; -\,(1-y) \, A^\beta{}_{\dot\beta} \star f_{\alpha\beta} \;-\; i(1-y)^2 \, A_{\beta\dot\beta} \star A^{\beta\dot\alpha} \star A_{\alpha\dot\alpha} \\ &\bar{D}_{\dot\beta} \, \tilde{\omega}_{\dot\alpha}| \;\rightarrow\; \bar{f}_{\dot\alpha\dot\beta} \;-\; i(1-y) \, A^\alpha{}_{\dot\alpha} \star A_{\alpha\dot\beta} \\ &D_\beta \, \bar{D}^2 \, \tilde{\omega}_{\dot\alpha}| \;\rightarrow\; -\,(1-y) \, \bar{f}_{\dot\alpha}{}^{\dot\beta} \star A_{\beta\dot\beta} \;+\; iy(1-y) \, A_{\alpha\dot\alpha} \star A^{\alpha\dot\beta} \star A_{\beta\dot\beta} \\ &\qquad -\; i(1-y) \, A_{\beta\dot\beta} \star A_{\alpha\dot\alpha} \star A^{\alpha\dot\beta} \;+\; i\partial_\beta{}^{\dot\beta} \, \bar{f}_{\dot\alpha\dot\beta} \;+\; (1-y) \, \partial_\beta{}^{\dot\beta} \, (A^\alpha{}_{\dot\alpha} \star A_{\alpha\dot\beta}) \end{aligned} \tag{4.6}$$

In the derivation of these components we made use of the Bianchi identities

$$\partial^{\beta}{}_{\dot\beta}\, A_{\beta\dot\alpha} \;+\; \partial^{\beta}{}_{\dot\alpha}\, A_{\beta\dot\beta} \;=\; -\,2\bar f_{\dot\alpha\dot\beta} \;+\; i\,[\, A^{\beta}{}_{\dot\alpha}\,,\, A_{\beta\dot\beta}\,]_{\star} \tag{4.7}$$

Proceeding in the same manner, for π_α and $\tilde\pi_{\dot\alpha}$ we find

$$\begin{aligned}
&\bar D_{\dot\beta}\,\pi_\alpha| \;\rightarrow\; i\,A_{\alpha\dot\beta}\\
&D_\beta \bar D^2\,\pi_\alpha| \;\rightarrow\; f_{\alpha\beta} \;-\; i(1-2y)\,A_{\beta}{}^{\dot\beta}\star A_{\alpha\dot\beta}\\
&D_\beta\,\tilde\pi_{\dot\alpha}| \;\rightarrow\; i\,A_{\beta\dot\alpha}\\
&D^2\bar D_{\dot\beta}\,\tilde\pi_{\dot\alpha}| \;\rightarrow\; \bar f_{\dot\alpha\dot\beta} \;+\; \partial^{\beta}{}_{\dot\beta}\,A_{\beta\dot\alpha} \;+\; iy\,A^{\beta}{}_{\dot\beta}\star A_{\beta\dot\alpha} \;-\; i(1-y)\,A^{\beta}{}_{\dot\alpha}\star A_{\beta\dot\beta}
\end{aligned} \tag{4.8}$$

We are now ready to compute the physical bosonic components contained in the supersymmetric anomaly (3.26). In that equation we perfom the superspace integration, which amounts to apply $D^2\bar D^2$ to the entire expression and evaluate everything at $\theta=\bar\theta=0$, and keep only bosonic terms.

The covariant term in eq. (3.26) gives

$$L \;\rightarrow\; \frac{i}{8\pi^2}\int d^4x\,\mathrm{Tr}(\,\lambda\, f^{\alpha\beta}\star f_{\alpha\beta}\,) \tag{4.9}$$

In the reduction of the consistent part only terms at the most linear in V contribute. Therefore, expanding up to the linear order, we are led to compute the expression

$$\begin{aligned}
\frac{i}{8\pi^2}\int d^4x\, D^2\,\bar D^2\Big\{\frac{1}{3}\int_0^1 dy\,y^2\,\mathrm{Tr}_s\Big(&\Lambda\star\pi^\alpha\star\omega_\alpha \;+\; (1-y)\,(V\,\Lambda)\star\pi^\alpha\star\omega_\alpha\\
&+\;(1-y)\,\Lambda\star\pi^\alpha\star(V\,\omega_\alpha)\;-\;(1-y)\,V\star(\tilde\pi^{\dot\alpha}\,\Lambda)\star\tilde\omega_{\dot\alpha}\Big)\Big\}
\end{aligned} \tag{4.10}$$

After a long but straightforward calculation, from each term in (4.10) we have

$$\begin{aligned}
&\frac{i}{8\pi^2}\int d^4x\,\frac{1}{3}\int_0^1 dy\,y^2\,D^2\bar D^2\,\mathrm{Tr}_s\,(\Lambda\star\pi^\alpha\star\omega_\alpha)\Big|\\
&\rightarrow -\frac{i}{8\pi^2}\,\mathrm{Tr}\Big[\,\frac{2}{3}\,\lambda\,f^{\alpha\beta}\star f_{\alpha\beta} - \frac{i}{12}\,\lambda\,\{A^{\alpha\dot\beta}\,,\,A^{\beta}{}_{\dot\beta}\star f_{\alpha\beta}\}_\star + \frac{i}{12}\,\lambda\,\{f^{\alpha\beta}\,,\,A_{\beta}{}^{\dot\alpha}\star A_{\alpha\dot\alpha}\}_\star\\
&\qquad +\;\frac{1}{30}\,\lambda\,\{A^{\alpha\dot\beta}\,,\,A_{\beta\dot\beta}\star A^{\beta\dot\alpha}\star A_{\alpha\dot\alpha}\}_\star \;+\; \frac{1}{60}\,\lambda\,\{A^{\beta\dot\beta}\star A^{\alpha}{}_{\dot\beta}\,,\,A_{\beta}{}^{\dot\alpha}\star A_{\alpha\dot\alpha}\}_\star\Big]
\end{aligned}$$

$$\begin{aligned}
&\frac{i}{8\pi^2}\int d^4x\,\frac{1}{3}\int_0^1 dy\,y^2\,(1-y)\,D^2\bar D^2\,\mathrm{Tr}_s\,((V\Lambda)\star\pi^\alpha\star\omega_\alpha)\Big|\\
&\rightarrow\;\frac{i}{8\pi^2}\,\mathrm{Tr}\Big[\,\frac{i}{12}\,\lambda\,\{A^{\alpha\dot\beta}\,,\,f_{\alpha}{}^{\beta}\}_\star\star A_{\beta\dot\beta} \;+\; \frac{1}{30}\,\lambda\,\{A^{\alpha\dot\beta}\,,\,A^{\beta\dot\alpha}\star A_{\alpha\dot\alpha}\}_\star\star A_{\beta\dot\beta}\,\Big]
\end{aligned}$$

$$\begin{aligned}
&\frac{i}{8\pi^2}\int d^4x\,\frac{1}{3}\int_0^1 dy\,y^2\,(1-y)\,D^2\bar D^2\,\mathrm{Tr}_s\,(\Lambda\star\pi^\alpha\star(V\,\omega_\alpha))\Big|\\
&\rightarrow\;-\,\frac{i}{8\pi^2}\,\mathrm{Tr}\Big[\,\frac{i}{12}\,\lambda\,\{A^{\alpha\dot\beta}\,,\,A^{\beta}{}_{\dot\beta}\star f_{\alpha\beta}\}_\star \;+\; \frac{1}{30}\,\lambda\,\{A^{\alpha\dot\beta}\,,\,A^{\beta}{}_{\dot\beta}\star A_{\beta}{}^{\dot\alpha}\star A_{\alpha\dot\alpha}\}_\star\,\Big]
\end{aligned}$$

$$-\frac{i}{8\pi^2}\int d^4x\,\frac{1}{3}\int_0^1 dy\,y^2\,(1-y)\,D^2\bar{D}^2\,\mathrm{Tr}_s\,(V\star(\tilde{\pi}^{\dot{\alpha}}\,\Lambda)\star\tilde{\omega}_{\dot{\alpha}})\Big|$$

$$\to\ \frac{i}{8\pi^2}\,\mathrm{Tr}\Big[\frac{i}{12}\,\lambda\,\{\bar{f}^{\dot{\alpha}\dot{\beta}}\,,\,A^{\beta}{}_{\dot{\beta}}\}_\star\star A_{\beta\dot{\alpha}}\ +\ \frac{1}{30}\,\lambda\,\{A^{\alpha\dot{\alpha}}\star A_{\alpha}{}^{\dot{\beta}}\,,\,A^{\beta}{}_{\dot{\beta}}\}_\star\star A_{\beta\dot{\alpha}}\Big] \quad (4.11)$$

The complete bosonic physical component is now obtained by adding (4.9), (4.11) and their complex conjugates. Using the identities [6]

$$\begin{aligned}\epsilon^{abcd} &= i[C^{\alpha\delta}C^{\beta\gamma}C^{\dot{\alpha}\dot{\beta}}C^{\dot{\gamma}\dot{\delta}}\ -\ C^{\alpha\beta}C^{\gamma\delta}C^{\dot{\alpha}\dot{\delta}}C^{\dot{\beta}\dot{\gamma}}]\\ F_{ab} &= \Big[C_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta}\ +\ C_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}}\Big]\quad,\quad \widetilde{F}_{ab}\ =\ i\Big[C_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta}\ -\ C_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}}\Big]\end{aligned} \quad (4.12)$$

where $F_{ab}\equiv\partial_a A_b-\partial_b A_a-i[A_a,A_b]_\star$ and $\widetilde{F}_{ab}\equiv\frac{1}{2}\epsilon_{abcd}F^{cd}$, the total result is

$$\begin{aligned}\mathcal{A}^{bos}_{\mathrm{BGJ}}(\lambda) &= \frac{1}{48\pi^2}\int d^4x\,\mathrm{Tr}\Big\{\lambda\Big[F_{ab}\star\widetilde{F}^{ab}\ +\ i\widetilde{F}^{ab}\star A_a\star A_b\\ &\quad +\ iA_a\star\widetilde{F}^{ab}\star A_b\ +\ iA_a\star A_b\star\widetilde{F}^{ab}\ -\ \epsilon^{abcd}A_a\star A_b\star A_c\star A_d\Big]\Big\}\end{aligned} \quad (4.13)$$

As a consequence of the identity

$$\epsilon^{abcd}\partial_a A_b\ =\ \frac{1}{2}\epsilon^{abcd}F_{ab}\ +\ i\epsilon^{abcd}A_a\star A_b \quad (4.14)$$

which follows from the definition of F_{ab}, the previous expression can be equivalently written as

$$\mathcal{A}^{bos}_{\mathrm{BGJ}}(\lambda)\ =\ \frac{1}{24\pi^2}\int d^4x\,\mathrm{Tr}\Big\{\lambda\,\epsilon^{abcd}\partial_a\Big(A_b\star\partial_c A_d\ -\ \frac{i}{2}A_b\star A_c\star A_d\Big)\Big\} \quad (4.15)$$

This result coincides with the bosonic consistent anomaly derived in [5].

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